TWO-LOOP EFFECTIVE ACTION FOR THEORIES WITH FERMIONS

B.A.FAIZULLAEV*
Physics Department, Middle East Technical University
Ankara 06531, Turkey
and
M.M.MUSAKHANOV
Theoretical Physics Department, Tashkent State University
Tashkent 700095, Uzbekistan

February 1, 2008

Abstract

On the basis of a new approach proposed in our previous work we develope a formalism for calculating of the effective action for some models containing fermion fields. This method allows us to calculate the fermionic part of the effective action (up to two-loop level) properly. The two-loop contribution to the effective potential for the Nambu-Jona-Lasinio model is calculated and is shown to vanish.

Number of pages: 25. Number of figures: 4.

*Permanent address: Theoretical Physics Department, Tashkent State University, Tashkent 700095, Uzbekistan

Running head: Two–loop effective action for theories with fermions

Name and the mailing address of the author to whom proofs should be sent:
Prof.Dr.Yousuf M.Musakhanov
Tashkent State University
Department of Theoretical Physics
700095 Tashkent, Uzbekistan.
tel: 7 3712 46 52 64
fax: 7 3712 39 45 16
E-Mail: yousuf@iceeat.silk.glas.apc.org yousuf@univer.tashkent.su

1 Introduction

The functional formulations of quantum field theories are the most attractive ones for investigation of general properties of these theories. The complete quantum-theoretical information can be extracted from the effective action functional. The effective action generates all 1PI Green's functions. But there are no special methods for the calculation of the effective action and usually the effective action is calculated either by direct summation of the infinite series of the Feynman diagrams [1] or by the functional integration method [2, 3]. In our previous works [4] we have developed a new method for the calculation of the effective action for quantum field models. This method is based on an old formula of DeWitt [5] that connects the vacuum expectation of a quantum functional with the classical ones (and its derivatives with respect to the classical fields).

The application of DeWitt formula leads to variational differential equations for effective action; by iterations over Planck constant $\hbar$ we can derive from them a loop expansion (for scalar theory this was done in [4]). Such computational method of course is simpler and more convenient compared to other methods [1, 2, 3] – we don't deal with the coefficients of each class of diagrams or with the picking out of nonacceptable diagrams as in [3].

By this method we have calculated the two - loop effective action for scalar $\lambda\varphi^4$ theory and for spinor electrodynamics in [4]. But in the last case our answer was not complete - we had not calculated fermionic part of the effective action. This work will be devoted to calculation of the complete effective action including the fermionic part of them for some fermionic models , namely, for QED and for bosonized Nambu-Jona-Lasinio (NJL) model. Fermionic part of the effective action is the part that generates amplitudes of processes with fermions at initial and/or final states.

The NJL model has a very attractive property: it manifests the dynamical acquiring of mass by fermion caused by spontaneous symmetry breaking [6, 7]. For investigation of the spontaneous symmetry breaking it is necessary to know the effective potential of the theory. The effective potential may be derived from the effective action by setting all the classical fields equals to constants. There are many works devoted to the calculation of the effective potential for different versions of the NJL model (see [8] and references therein), but usually almost all authors are dealing with one-loop approximation or some asymptotics of the potential. In this work we calculate the effective potential for the NJL model at the two-loop level for two cases: two- and four-dimensional space-time. We show that the two-loop contribution to the effective potential vanishes for the both cases. Thus, the dynamical spontaneous symmetry breaking property of the NJL model is determined only by the one-loop correction.

The article is organized as follows: in Sec.2 we derive the DeWitt formula for scalar and for spinor theories; in Sec.3 we describe the relationships between the classical and the effective actions; in Sec.4 we calculate the two–loop effective action for scalar $\lambda\varphi^4$ theory and in Sec.5 we describe a formal method of integrating of the equations. In

Sec.6 we introduce a new entity-the functional which is the effective action involving only one variable and is the generating functional of the connected Green's functions with respect to another variables. This functional is analogous to the well known Rauss function in classical dynamics. Introduction of this functional permits us to calculate amplitudes of any processes with fermions at initial and/or final states. In the remaining part of the Sec.6 we calculate the two–loop effective action for QED and in Sec.7 the two-loop effective action for the bosonized NJL model. In Sec.8 we will show that the two–loop contribution to the effective potential of the NJL model vanishes.

2 DeWitt's formula

Let's begin with scalar case and define the generating functional of Green's functions as:

$$Z[J] = \exp\left(iW[J]\right) = \int D\Phi \exp\left(i \int (L + J\Phi)\, dx\right), \tag{1}$$

where $W[J]$ is generating functional of the connected Green's function and L is is the Lagrangian for scalar field $\Phi(x)$. Consider the following expression:

$$\begin{aligned} \exp\left(iW[J+\zeta]\right) &= \int D\Phi \exp\{i \int (L + (J+\zeta)\Phi)\, dx\} = \\ &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \zeta_{i_1}\zeta_{i_2}\cdots\zeta_{i_n} \int D\Phi \Phi_{i_1}\Phi_{i_2}\ldots\Phi_{in} \exp\{i \int (L + J\mathit{\Phi})\, dx\}, \end{aligned} \tag{2}$$

where we use the condensed notations $\Phi_i = \Phi(x_i), \Phi_i\zeta_i = \int \Phi(x)\zeta(x)dx$. If $\hat{\varphi}$ is a quantum field then we may define vacuum expectation value of any quantum functional $Q[\hat{\varphi}]$ as follows

$$\langle Q[\hat{\varphi}]\rangle = \exp\left(-iW[J]\right) \int D\Phi Q[\Phi] \exp\left(i \int (L + J\Phi)\, dx\right).$$

In this formula $Q[\Phi]$ in the integrand is the classical functional, that is the functional depending on c-functions $\Phi(x)$. Let's expand $Q[\Phi]$ in the integrand in the Taylor series:

$$\begin{aligned} \langle Q[\hat{\varphi}]\rangle &= \exp\left(-iW[J]\right) \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\delta^n W}{\delta\Phi_1^i \delta\Phi_2^i \cdots \delta\Phi_n^i}_{\Phi=0} \cdot \\ &\int D\Phi \Phi_{i_1}\Phi_{i_2}\ldots\Phi_{in} \exp\{i \int (L + J\mathit{\Phi})\, dx\}, \end{aligned} \tag{3}$$

Comparing Eq.(2) and Eq.(3) we get:

$$\langle Q[\hat{\varphi}]\rangle =: \exp\left(-iW[J]\right) \exp\left(iW[J + \frac{\delta}{i\delta\Phi}]\right) : Q[\Phi]_{\Phi=0}, \tag{4}$$

where the colons mean that derivatives acts only on $Q[\Phi]$. Substituting the following expansion

$$W[J+\zeta] = W[J] + \zeta_i \frac{\delta W}{\delta J_i} + \sum_{n=2}^{\infty} \frac{1}{n!} \zeta_{i_1} \zeta_{i_2} \cdots \zeta_{i_n} G^{i_1 i_2 \cdots i_n},$$

where

$$G^{i_1 i_2 \cdots i_n} = \frac{\delta^n W}{\delta J_{i_1} \delta J_{i_2} \cdots \delta J_{i_n}}$$

is connected Green's functions, into Eq.(4) we get the following formula of DeWitt ([5], ch.22):

$$\langle Q[\hat{\varphi}] \rangle =: \exp\left(\frac{i}{\hbar} \sum_{n=2}^{\infty} \frac{(-i\hbar)^n}{n!} G^{i_1 i_2 \ldots i_n} \frac{\delta^n}{\delta \varphi^{i_1} \delta \varphi^{i_2} \ldots \delta \varphi^{i_n}} \right) : Q[\varphi] \tag{5}$$

where the φ in the rhs of this equation means the so-called classical fields:

$$\varphi_i = \delta W[J]/\delta J_i. \tag{6}$$

Since we will deal with the loop expansion we have restored the Planck constant $\hbar$ in DeWitt's formula.

For the models containing fermion and vector fields the generalization of DeWitt's formula leads to the following expression:

$$\langle Q[\hat{A}_\mu, \hat{\psi}, \hat{\bar{\psi}}] \rangle =: \exp\left(\hat{G} \right) : Q[A_\mu, \psi, \bar{\psi}] \tag{7}$$

where the operator $\hat{G}$ is defined as follows:

$$\hat{G} = \frac{i}{\hbar} \sum_{n=2}^{\infty} \frac{(-i\hbar)^n}{n!} \sum C^n_{ijk} (-1)^j \cdot$$

$$\cdot G^{\mu_1 \ldots \mu_i \alpha_1 \ldots \alpha_j \beta_1 \ldots \beta_k} \frac{\delta^n}{\delta A_{\mu_1} \ldots \delta A_{\mu_i} \delta \psi_{\beta_k} \ldots \delta \psi_{\beta_1} \delta \bar{\psi}_{\alpha_j} \ldots \delta \bar{\psi}_{\alpha_1}}.$$

The classical fields are defined as follows:

$$A_\mu = \delta W/\delta J_\mu, \qquad \psi = \delta W/\delta \bar{\eta}, \qquad \bar{\psi} = -\delta W/\delta \eta.$$

The coefficients C^n_{ijk} are defined as

$$C^n_{ijk} = n!/(i!j!k!), \qquad i+j+k = n,$$

and

$$G^{\mu_1 \ldots \mu_i \alpha_1 \ldots \alpha_j \beta_1 \ldots \beta_k} = \delta^n W / \left(\delta J_{\mu_1} \ldots \delta J_{\mu_i} \delta \eta_{\beta_k} \ldots \delta \eta_{\beta_1} \delta \bar{\eta}_{\alpha_j} \ldots \delta \bar{\eta}_{\alpha_1} \right)$$

- are the connected Green's function with i photon and $j+k$ fermion legs. Therefore, the vacuum expectation value of a quantum functional is defined by the (classical) functional and derivatives of them where quantum fields are replaced by classical fields.

3 The Relations Between Classical and Effective Actions

Usually the effective action $\Gamma[\varphi]$ is defined as

$$\Gamma[\varphi] = W[J] - J_i \varphi^i, \tag{8}$$

where $\varphi_i = \delta W[J]/\delta J_i$; but for our purposes a more convenient definition is that of DeWitt [5]:

$$\langle \frac{\delta S}{\delta \hat{\varphi}_i} \rangle = \frac{\delta \Gamma}{\delta \varphi_i}. \tag{9}$$

This definition is a consequence of the following well known relations:

$$\frac{\delta \Gamma}{\delta \varphi^i} = -J_i, \qquad \langle \frac{\delta S}{\delta \hat{\varphi}^i} \rangle = -J_i. \tag{10}$$

The last of these is the "quantum equations of motion" and the first is consequence of Eq.(8). Comparing of Eq.(5) and Eq.(9) leads to the main formula:

$$\frac{\delta \Gamma}{\delta \varphi^i} =: \exp\{\frac{i}{\hbar} \sum_{n=2}^{\infty} \frac{(-i\hbar)^n}{n!} G^{i_1 i_2 \cdots i_n} \frac{\delta^n}{\delta \varphi_1^i \delta \varphi_2^i \cdots \delta \varphi_n^i} \} : \frac{\delta S}{\delta \varphi^i}. \tag{11}$$

This formula in principle, permits us by using the known classical action S to restore the quantum effective action Γ. We need only an additional relation between Γ and G^{ij}. Clearly this relation and Eq.(11) will have different forms for different theories.

For the spinor electrodynamics

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi} \left[i \left(\hat{\partial} - ie\hat{A} \right) + m \right] \Psi - \frac{1}{2\alpha} \left(\partial_\mu A^\mu \right)^2 \tag{12}$$

we may write relations connecting classical and effective actions as follows:

$$\frac{\delta \Gamma}{\delta A_\mu} = \langle \frac{\delta S}{\delta \hat{A}_\mu} \rangle, \qquad \frac{\delta \Gamma}{\delta \bar{\psi}^\alpha} = \langle \frac{\delta S}{\delta \hat{\bar{\psi}}^\alpha} \rangle, \qquad \frac{\delta \Gamma}{\delta \psi^\alpha} = \langle \frac{\delta S}{\delta \hat{\psi}^\alpha} \rangle. \tag{13}$$

These formulas are most convenient for application of Eqs.(7) and (9).

4 Effective Action for Scalar $\lambda\varphi^4/4!$ Theory

Let's take the classical action in the following form:

$$S = -\frac{1}{2} \varphi (\partial^2 + m^2) \varphi - \lambda \varphi^4 / 4!. \tag{14}$$

Since

$$\frac{\delta S}{\delta \varphi^i} = -(\partial^2 + m^2) \varphi_i - \lambda \varphi_i^3 / 6,$$

then in Eq.(11) leaves finite number of terms:

$$\frac{\delta\Gamma}{\delta\varphi^i} = iK^{-1}_{ij}\varphi^j - \frac{\lambda}{6}\varphi_i^3 + \frac{1}{2}i\lambda\hbar G^{ii}\varphi^i + \frac{\lambda}{6}\hbar^2 G^{ij}G^{ik}G^{il}\frac{\delta^3\Gamma}{\delta\varphi^j\delta\varphi^k\delta\varphi^l}. \tag{15}$$

This is (variational) differential equations for the effective action $\Gamma[\varphi]$ [4]. Here

$$K_{ij} = -i(\partial^2 + m^2)^{-1}\delta_{ij}$$

is the free Feynman Green's function and in third and fourth terms there are no summations over i. Differentiating the first of Eq.(10) with respect to J_k we get

$$G^{ij}\frac{\delta^2\Gamma}{\delta\varphi^j\delta\varphi^k} = -\delta^i_k. \tag{16}$$

Equations (15) and (16) forms the system of equations for the calculation of the effective action for the scalar theory. We would like to derive the loop expansion; thus we will iterate equations (15) and (16) over $\hbar$.

Let's expand Γ and G over $\hbar$:

$$\begin{aligned} \Gamma &= \Gamma_0 + \hbar\Gamma_1 + \hbar^2\Gamma_2 + \cdots, \\ G &= G_0 + \hbar G_1 + \hbar^2 G_2 + \cdots, \end{aligned} \tag{17}$$

where $\Gamma_0, \Gamma_1, \Gamma_2, \ldots$–corresponds to the tree, the one–loop, the two–loop , etc., approximations, respectively. This is also true for $G_0, G_1, G_2, \ldots$, etc.,respectively. The substitution of Eq.(17) to Eq.(16) permits us rewrite Eq.(16) in a more convenient form for the loop expansion:

$$G_0^{ij}\frac{\delta^2\Gamma_0}{\delta\varphi^j\delta\varphi^k} = -\delta^i_k, \qquad G_1^{ij}\frac{\delta^2\Gamma_0}{\delta\varphi^j\delta\varphi^k} + G_0^{ij}\frac{\delta^2\Gamma_1}{\delta\varphi^j\delta\varphi^k} = 0, \ldots \tag{18}$$

From Eq.(15) we have the following equation for Γ_0:

$$\frac{\delta\Gamma_0}{\delta\varphi^i} = iK^{-1}_{ij}\varphi^j - \frac{\lambda}{6}\varphi_i^3. \tag{19}$$

This equation gives us:

$$\Gamma_0 = \frac{1}{2}i\varphi^i K^{-1}_{ij}\varphi^j - \frac{\lambda}{24}(\varphi^i)^4, \tag{20}$$

or, in the usual notations

$$\Gamma_0 = \frac{i}{2}\int \varphi(x)K^{-1}(x-y)\varphi(y)dxdy\frac{\lambda}{4!}\int \varphi^4(x)dx = S.$$

It is suitable to introduce the following notation:

$$\Phi_{ij} = iK^{-1}_{ij} - \frac{\lambda}{2}(\varphi^i)^2\delta_{ij} = iK^{-1}_{ik}\left(1 + i\frac{\lambda}{2}K\varphi^2\right)_{kj}. \tag{21}$$

Then from the first of Eqs.(18) we get:

$$G_0^{ij} = -(\Phi^{-1})^{ij}. \tag{22}$$

Now we may to calculate the Γ_1. First note that:

$$\frac{\delta\Gamma_1}{\delta\varphi^i} = i\frac{\lambda}{2}G_0^{ii}\varphi^i = -i\frac{\lambda}{2}(\Phi^{-1})^{ii}\varphi^i, \tag{23}$$

which can be rewritten as follows:

$$\delta\Gamma_1 = \frac{i}{2}(\Phi^{-1})^{ij}\delta\Phi_{ji}. \tag{24}$$

The integration of Eq.(24) gives us the well known one–loop result (after the second equality sign we are dropping the trivial infinite term):

$$\Gamma_1 = \frac{i}{2}\mathrm{Tr}\,\log\Phi = \frac{i}{2}\mathrm{Tr}\,\log\left(1 + i\frac{\lambda}{2}K\varphi^2\right). \tag{25}$$

We get for Γ_2 :

$$\frac{\delta\Gamma_2}{\delta\varphi^i} = i\frac{\lambda}{2}G_1^{ii}\varphi^i + \frac{\lambda}{6}G_0^{ij}G_0^{ij}G_0^{ik}G_0^{il}\frac{\delta^3\Gamma_0}{\delta\varphi^j\delta\varphi^k\delta\varphi^l}$$

(on the r.h.s. of the equation there are no summations over i). Determining G_1 from the second of the Eq.(18) we find:

$$\frac{\delta\Gamma_2}{\delta\varphi^i} = \frac{\lambda^2}{4}(\Phi^{-1})^{is}(\Phi^{-1})^{ss}(\Phi^{-1})^{si}\varphi^i +$$

$$+\frac{\lambda^3}{4}(\Phi^{-1})^{il}(\Phi^{-1})^{kl}\varphi^l(\Phi^{-1})^{lk}\varphi^k(\Phi^{-1})^{ki}\varphi^i + \frac{\lambda^2}{6}((\Phi^{-1})^{il})^3\varphi^l$$

(there are no summations over i). After some manipulations we get [4]:

$$\Gamma_2 = \frac{\lambda}{8}((\Phi^{-1})^{ii})^2 + \frac{\lambda^2}{12}\varphi^i((\Phi^{-1})^{ij})^3\varphi^j. \tag{26}$$

To this expression corresponds the set of diagrams depicted in Fig.1.

Figure 1.

5 Effective action for QED

Application of our main Eq.(7) to Eqs.(13) give rise to the following system of equations :

$$\frac{\delta\Gamma}{\delta A_\mu} = e\bar{\psi}\gamma^\mu\psi + D^{-1\mu\nu}A_\nu + ie\hbar\mathrm{Tr}\left(\gamma^\mu\frac{\delta^2 W}{\delta\eta\delta\bar{\eta}}\right), \tag{27}$$

$$\frac{\delta\Gamma}{\delta\bar{\psi}^\alpha} = \Big((i\hat{\partial} + e\hat{A} - m)\psi\Big)^\alpha - ie\hbar\,(\gamma_\mu)^{\alpha\beta}\,\frac{\delta^2\,W}{\delta J_\mu\delta\bar{\eta}^\beta}, \tag{28}$$

$$\frac{\delta\Gamma}{\delta\psi^\alpha} = \left(\bar{\psi}(i\overleftarrow{\hat{\partial}} - e\hat{A} + m)\right)^\alpha - ie\hbar\,(\gamma_\mu)^{\beta\alpha}\,\frac{\delta^2 W}{\delta J_\mu\delta\eta^\beta}. \tag{29}$$

where

$$D^{\mu\nu} = \frac{1}{\partial^2}\left(g^{\mu\nu} - (1-\alpha)\frac{\partial^\mu\partial^\nu}{\partial^2}\right)$$

is the free photon Green's function.

The system of equations (27)-(28)-(29) is not complete - to those we must add equations relating Γ to the connected Green's functions G. For the spinor model the relations between Γ and G are very complicated, and for this reason we developed a new formalism (which follows).

Let's introduce the following notations for the classical fields and the sources:

$$\alpha = (A_\mu, -\psi, \bar{\psi}), \qquad \beta = (J_\mu, \bar{\eta}, \eta), \tag{30}$$

and also

$$\Gamma_{i_1 i_2\cdots i_n} = \frac{\delta^n\Gamma}{\delta\alpha^{i_1}\delta\alpha^{i_2}\cdots\delta\alpha^{i_n}}, \qquad G^{ij\cdots n} = \epsilon_{i_1}\cdots\epsilon_{i_n}\frac{\delta^n W}{\delta\beta_{i_1}\delta\beta_{i_2}\cdots\delta\beta_{i_n}}, \tag{31}$$

where ϵ_i is equal to $+1$ if $i=\mu$ and -1 if otherwise. In these notations the index i may take the values μ, α and $\bar{\alpha}$, where $\alpha(\bar{\alpha})$ correspond to the index of the field $\psi^\alpha(\bar{\psi}^\alpha)$. Then our equations (27-28-29) can be written in the form:

$$\Gamma_i = S_i + ie\hbar\gamma_{ijk}G^{kj} \tag{32}$$

where

$$\gamma_{ijk} = \begin{cases} \gamma_\mu^{\alpha\bar{\beta}} & \text{if } i=\mu, \\ \gamma_\mu^{\bar{\beta}\alpha} & \text{if } i=\alpha, \\ \gamma_\mu^{\bar{\alpha}\beta} & \text{if } i=\bar{\alpha}. \end{cases} \tag{33}$$

In this notations the equation for Γ have the following form

$$\Gamma_i = -\beta_i.$$

Differentiating it with respect to β_j we may derive relation between Γ and $G's$:

$$G^{ij}\Gamma_{jk} = -\epsilon_i\delta^i_k. \tag{34}$$

Now we may rewrite our equations as follows:

$$\Gamma_i = S_i - ie\hbar\gamma_{ijk}\epsilon_j \left(\hat{\Gamma}^{-1}\right)^{kj} \tag{35}$$

where $\hat{\Gamma}^{-1}$ is the inverse matrix of $\{\Gamma_{ij}\}$. The solution of this system of equations (35) provide us the effective action for QED. Next we will present a formal approach for solving this system.

Expanding Γ in terms of $\hbar$ as in Eq.(17) we obtain for Γ_0:

$$\Gamma_{0i} = S_i.$$

That is, we have

$$\Gamma_0 = S, \tag{36}$$

as it must be. For Γ_1 we have

$$\Gamma_{1i} = -ie\gamma_{ijk}\epsilon_j \left(\hat{S}^{-1}\right)^{kj}. \tag{37}$$

In other words

$$\delta\Gamma_1 = -ie\gamma_{ijk}\delta\alpha^i\epsilon_j \left(\frac{\delta^2 S}{\delta\alpha^j\delta\alpha^k}\right)^{-1}. \tag{38}$$

If we return to usual notations we have:

$$\begin{aligned} \delta\Gamma_1 = & -ie\delta A^\mu \mathrm{Tr}\left[\gamma_\mu \left(\tfrac{\delta^2 S}{\delta\psi\delta\bar{\psi}}\right)^{-1}\right] + \\ & +ie\left[\delta\bar{\psi}\gamma_\mu \left(\tfrac{\delta^2 S}{\delta\bar{\psi}\delta A_\mu}\right)^{-1} - \left(\tfrac{\delta^2 S}{\delta\psi\delta A_\mu}\right)^{-1}\gamma_\mu\delta\psi\right]. \end{aligned} \tag{39}$$

The first term of Eq.(39) can be easily integrated to yield:

$$\Gamma_1^A = -i\mathrm{Tr}\,\log\left(i\hat{\partial} + e\hat{A} - m\right). \tag{40}$$

For the non-fermionic part of Γ we can compute also the two-loop contribution without serious difficulties [4]. For this we neglect all the terms with fermions in initial and/or final states and get for non-fermionic part of the Γ:

$$\frac{\delta\Gamma^A}{\delta A_\mu} = D^{-1\mu\nu}A_\nu - ie\hbar\mathrm{Tr}\left(\hat{K}\gamma^\mu\right) - e^2\hbar^2\mathrm{Tr}\left(G^\nu\gamma_\nu\hat{K}\gamma^\mu\right), \tag{41}$$

where G^ν is three–point connected Green's function with one photon and two electron legs and

$$\hat{K}^{-1} = i\hat{\partial} + e\hat{A} - m.$$

The last term of Eq.(41) consist of two–, three–, and etc. loops contributions to Γ^A. For example, the two–loops contribution to $\delta\Gamma^A/\delta A_\mu$ has the following form:

$$e^3\hbar^2\mathrm{Tr}\left(\gamma^\mu\hat{K}\gamma_\sigma\hat{K}\gamma_\nu\hat{K}\right)D^{\sigma\nu},$$

from which we get:
$$\delta\Gamma_2^A = -e^2 \mathrm{Tr}\left(\delta\hat{K}\gamma_\sigma\hat{K}\gamma_\nu\right) D^{\sigma\nu}.$$
Hence,
$$\Gamma_2^A = -\frac{1}{2}e^2 \mathrm{Tr}\left(\gamma_\mu\hat{K}\gamma_\nu\hat{K}\right) D^{\mu\nu}. \tag{42}$$
The first term of this expression (if we expand $\hat{K}$ over $\hat{A}$) is the vacuum polarization diagram, other terms contribute to the amplitudes with only photons in the external legs.

But the calculation of the fermionic part of Γ is a rather difficult problem. Due to this, we will use another method to solve this problem in the following section .

6 Fermionic Part of the Effective Action

Let us define the generating functional
$$Z[J_\mu,\eta,\bar{\eta}] = \int DA_\mu D\psi D\bar{\psi} \exp\left(i\int\left(L + JA_\mu + \bar{\eta}\psi + \bar{\psi}\eta\right)dx\right). \tag{43}$$
The integrations over spinor variables leads us to an effective generating functional:
$$Z[J_\mu,\eta,\bar{\eta}] = \int DA_\mu \exp\left(iS_{eff}\right), \tag{44}$$
where
$$S_{eff} = S_A - F + i\hbar T, \tag{45}$$
and:
$$\begin{gathered} S_A = \tfrac{1}{2}A_\mu D^{-1}_{\mu\nu}A_\nu, \qquad F = \bar{\eta}\hat{K}\eta, \qquad T = \mathrm{Tr}\log\hat{K}, \\ \hat{K}^{-1} = i\hat{\partial} + e\hat{A} - m, \qquad D_{\mu\nu} = \frac{1}{\partial^2}\left(g_{\mu\nu} - (1-\alpha)\frac{\partial_\mu\partial_\nu}{\partial^2}\right). \end{gathered} \tag{46}$$
Let's define the following "effective action"
$$\tilde{\Gamma}[A_\mu,\eta,\bar{\eta}] = W[J_\mu,\eta,\bar{\eta}] - J_\mu A^\mu \tag{47}$$
which is the effective action in terms of the A_μ variable and generates connected Green's functions with respect to η and $\bar{\eta}$ variables. So, we perform Legendre transformations only with respect to J_μ variables without disturbing $(\eta,\bar{\eta})$ variables. The new functional $\tilde{\Gamma}$ is analogous to the Rauss function in classical mechanics. By the application of the DeWitt's formula to S_{eff} we get the following expression:
$$\frac{\delta\tilde{\Gamma}}{\delta A_\mu} =: \exp(\hat{G}) : \frac{\delta S_{eff}}{\delta A_\mu}, \tag{48}$$

where

$$\hat{G} = \frac{-i\hbar}{2} G^{\mu\nu} \frac{\delta^2}{\delta A^\mu \delta A^\nu} - \frac{\hbar^2}{6} G^{\mu\nu\lambda} \frac{\delta^3}{\delta A^\mu \delta A^\nu \delta A^\lambda} + \ldots, \tag{49}$$

and

$$G^{\mu\nu\ldots\lambda} = \frac{\delta^n W}{\delta A_\mu \delta A_\nu \ldots \delta A_\lambda}.$$

After substituting the following formula

$$\frac{\delta S_{eff}}{\delta A_\mu} = D^{-1\mu\nu} A_\nu + F^\mu - i\hbar T^\mu, \tag{50}$$

into Eq.(48), and introducing

$$F^{\mu\nu\ldots\lambda} = (-1)^s \frac{\delta^s F}{\delta A_\mu \delta A_\nu \ldots \delta A_\lambda}, \qquad T^{\mu\nu\ldots\lambda} = (-1)^s \frac{\delta^s T}{\delta A_\mu \delta A_\nu \ldots \delta A_\lambda} \tag{51}$$

we get

$$\begin{aligned} \delta\tilde{\Gamma}/\delta A_\mu = D^{-1\mu\nu} A_\nu + F^\mu - i\hbar T^\mu - \tfrac{i}{2}\hbar G^{\sigma\lambda} \left(F_{\sigma\lambda\mu} - i\hbar T_{\sigma\lambda\mu}\right) - \\ -\hbar^2 [\tfrac{1}{6} G^{\sigma\lambda\nu} \left(-F_{\sigma\lambda\nu\mu} + i\hbar T_{\sigma\lambda\nu\mu}\right) + \tfrac{1}{8} G^{\sigma\lambda} G^{\nu\rho} \left(F_{\sigma\lambda\nu\rho\mu} - i\hbar T_{\sigma\lambda\nu\rho\mu}\right)] + \ldots \end{aligned} \tag{52}$$

Expanding G and $\tilde{\Gamma}$ in terms of $\hbar$ as in Eq.(17) we get:

$$\frac{\delta\tilde{\Gamma}_0}{\delta A_\mu} = D^{-1\mu\nu} A_\nu + F^\mu, \tag{53}$$

which provides us

$$\tilde{\Gamma}_0 = \frac{1}{2} A_\mu D^{-1\mu\nu} A_\nu - \hat{F} = S. \tag{54}$$

Using

$$G^{\mu\nu} \frac{\delta^2 \tilde{\Gamma}}{\delta A^\nu \delta A^\lambda} = -\delta^\mu_\lambda \tag{55}$$

or, in other words

$$G_0^{\mu\nu} \frac{\delta^2 \tilde{\Gamma}_0}{\delta A^\nu \delta A^\lambda} = -\delta^\mu_\lambda, \qquad G_1^{\mu\nu} \frac{\delta^2 \tilde{\Gamma}_0}{\delta A^\nu \delta A^\lambda} + G_0^{\mu\nu} \frac{\delta^2 \tilde{\Gamma}_1}{\delta A^\nu \delta A^\lambda} = 0 \ldots \tag{56}$$

we get:

$$G_0^{\mu\nu} \left(D^{-1}_{\nu\lambda} - F_{\nu\lambda}\right) = -\delta^\mu_\lambda,$$

or, in the matrix notation

$$\hat{G}_0 = -\hat{D} \left(I - \hat{F}\hat{D}\right)^{-1}. \tag{57}$$

The last matrix equation must be understand as follows:

$$G^\mu_{0\nu} = -D^\mu_\sigma \left(\delta^\sigma_\nu + F^{\sigma\lambda} D_{\lambda\nu} + F^{\sigma\lambda} D_{\lambda\rho} F^{\rho\alpha} D_{\alpha\nu} + \ldots\right).$$

For $\tilde{\Gamma}_1$ we get the equation

$$
\begin{aligned}
\frac{\delta \tilde{\Gamma}_1}{\delta A_\mu} &= -iT^\mu - \frac{i}{2} G_0^{\sigma\lambda} F^\mu_{\sigma\lambda} = i \frac{\delta}{\delta A_\mu} T + \frac{i}{2} G_0^{\sigma\lambda} \frac{\delta}{\delta A_\mu} F_{\sigma\lambda} = \\
&= \frac{\delta}{\delta A_\mu} [iT + \frac{i}{2} \mathrm{Tr}\ \log(I - \hat{F}\hat{D})],
\end{aligned}
\tag{58}
$$

which after integration gives us:

$$
\begin{aligned}
&\tilde{\Gamma}_1 = iT + \tfrac{i}{2} \mathrm{Tr}\ \log(I - \hat{F}\hat{D}) = \\
&= i\mathrm{Tr}\ \log \hat{K} + \tfrac{i}{2} \mathrm{Tr}\ \log \left(1 - e^2 \left(\bar{\eta}\hat{K}\gamma_\mu \hat{K}\gamma_\nu \hat{K}\eta + \bar{\eta}\hat{K}\gamma_\nu \hat{K}\gamma_\mu \hat{K}\eta\right) D^{\mu\nu}\right).
\end{aligned}
\tag{59}
$$

The second term of the Eq.(59) represents the one-loop fermionic part of the "effective action" $\tilde{\Gamma}$. By differentiation of $\tilde{\Gamma}$ we may compute any (one-loop) amplitudes with fermion legs.

The equation for $\tilde{\Gamma}_2$ has the form

$$
\frac{\delta \tilde{\Gamma}_2}{\delta A_\mu} = -\frac{i}{2} G_1^{\sigma\lambda} F^\mu_{\sigma\lambda} - \frac{1}{2} G_0^{\sigma\lambda} T^\mu_{\sigma\lambda} + \frac{1}{6} G_0^{\sigma\lambda\nu} F^\mu_{\sigma\lambda\nu} - \frac{1}{8} G_0^{\sigma\lambda} G_0^{\nu\rho} F^\mu_{\sigma\lambda\nu\rho} \tag{60}
$$

For solving this equation we must first calculate G_1 from Eq.(56). As shown in Appendix A G_1 is:

$$
G_1^{\mu\nu} = G_0^{\mu\sigma} \left(iT_{\sigma\lambda} - \frac{i}{2} \frac{\delta}{\delta A^\sigma} \left(G_0^{\alpha\beta} F^\lambda_{\beta\alpha} \right) \right).
$$

and then the solution of the Eq.(60) is

$$
\tilde{\Gamma}_2 = \frac{1}{2} \mathrm{Tr}\, (\hat{G}_0 \hat{T}) + \frac{1}{8} G_0^{\alpha\beta} G_0^{\gamma\delta} F_{\alpha\beta\gamma\delta} - \frac{1}{12} G_0^{\alpha\sigma} G_0^{\beta\nu} G_0^{\gamma\lambda} F_{\alpha\beta\gamma} F_{\sigma\nu\lambda} \tag{61}
$$

Each term of this expression may be represented in graphic form as in Figs.2,3 and 4.

Figure 2.

Figure 3.

Figure 4.

The expressions for $\tilde{\Gamma}_0, \tilde{\Gamma}_1, \tilde{\Gamma}_2$ of course, have a symbolic sense; we have determined the algebraic structure of the effective action and each term of these expressions is a complicated integral. For derivation of the explicit formulas for effective action, all the integrations and trace calculations must be performed. Such a program can be carried out with specified functions A_μ, ψ etc. in integrands.

7 The Effective Action for the NJL model

We would like to calculate the effective action for one of the widely discussed models-the model of Nambu-Jona-Lasinio. We take the model in the following bosonized form:

$$L = \bar{\psi}(i\hat{\partial} + g(\sigma + i\gamma_5\pi))\psi - \frac{1}{2}(\sigma^2 + \pi^2) \tag{62}$$

where σ, π are auxiliary scalar and pseudoscalar fields.

The equations for effective action for this model are

$$\begin{aligned} \frac{\delta\Gamma}{\delta\phi^i} &= -\phi^i + g\bar{\psi}\gamma_i\psi + g\mathrm{Tr}\left(\gamma_i \frac{\delta^W}{\delta\eta\delta\bar{\eta}}\right), \\ \frac{\delta\Gamma}{\delta\psi} &= \bar{\psi}(i\overleftarrow{\hat{\partial}} - g\hat{\phi}) - ig\hbar\gamma_i\frac{\delta^2 W}{\delta\eta\delta J_i}, \\ \frac{\delta\Gamma}{\delta\bar{\psi}} &= (i\hat{\partial} + g\hat{\phi})\psi - ig\hbar\gamma_i\frac{\delta^2 W}{\delta\bar{\eta}\delta J_i}. \end{aligned} \tag{63}$$

Here we used the notations

$$\hat{\phi} = \phi^i\gamma^i, \qquad \phi^i = \{\sigma, \pi\}, \qquad \gamma^i = \{1, i\gamma_5\}. \tag{64}$$

This system of equations may be solved by iterations with respect to either g or $\hbar$. But instead here we adopt the method used earlier for QED.

The generating functional for this model can be written as

$$\begin{aligned} Z[J,\eta,\bar{\eta}] &= \int D\phi D\bar{\psi}D\psi \exp i\left(L + J_i\phi^i + \bar{\psi}\eta + \bar{\eta}\psi\right) = \\ &= \int D\phi \exp\left(iS_{eff} + iJ_i\phi^i\right), \end{aligned} \tag{65}$$

where we have integrated out the spinor variables and have introduced the following notations

$$\begin{aligned} &S_{eff} = -\tfrac{1}{2}\phi^2 - F + i\hbar T, \qquad \phi^2 = \sigma^2 + \pi^2, \\ &F = \bar{\eta}\hat{K}\eta, \qquad T = \mathrm{Tr}\,\log\hat{K}, \qquad \hat{K}^{-1} = i\hat{\partial} + g\hat{\phi}. \end{aligned} \tag{66}$$

The free propagator for the auxiliary scalar field ϕ^i is

$$D_{ij}(x-y) = \delta(x-y)\delta_{ij} \tag{67}$$

and S_{eff} may be represented in the following form

$$S_{eff} = -\frac{1}{2}\phi^i D_{ij}^{-1}\phi^j - F + i\hbar T. \tag{68}$$

Then for $\tilde{\Gamma}$ we have the following equation

$$\frac{\delta\tilde{\Gamma}}{\delta\phi^i} =: \exp\left(\frac{i}{\hbar}\sum_{n=2}^{\infty}\frac{(-i\hbar)^n}{n!}G^{i_1\cdots i_n}\frac{\delta^n}{\delta\phi^{i_1}\cdots\delta\phi^{i_n}}\right) : \frac{\delta S_{eff}}{\delta\phi^i}. \tag{69}$$

Using the notations

$$F_{i_1\cdots i_n} = (-1)^n\frac{\delta^n}{\delta\phi^{i_1}\cdots\delta\phi^{i_n}}F, \qquad T_{i_1\cdots i_n} = (-1)^n\frac{\delta^n}{\delta\phi^{i_1}\cdots\delta\phi^{i_n}}T, \tag{70}$$

we have for the derivative of $\tilde{\Gamma}$

$$\begin{aligned}\frac{\delta\tilde{\Gamma}}{\delta\phi^i} &= -D_{ij}^{-1}\phi^j + F_i - i\hbar T_i - \frac{i}{2}\hbar G^{lj}\left(F_{ijl} - i\hbar T_{ijl}\right) - \\ &-\hbar^2\left[\frac{1}{6}G^{jkl}\left(-F_{lkji} + i\hbar T_{lkji}\right) - \frac{1}{8}G^{lj}G^{ks}\left(F_{jlksi} - i\hbar T_{jlksi}\right)\right] + \ldots .\end{aligned} \tag{71}$$

Expanding the connected Green's functions G^{ij} in terms of $\hbar$, we have

$$\begin{aligned}\frac{\delta\tilde{\Gamma}}{\delta\phi^i} &= -D_{ij}^{-1}\phi^j + F_i + \hbar\left[-iT_i - \frac{i}{2}G_0^{lj}F_{jli}\right] + \\ &+\hbar^2\left[-\frac{i}{2}G_1^{lj}F_{jli} - \frac{1}{2}G_0^{lj}T_{jli} + \frac{1}{6}G_0^{jkl}F_{lkji} - \frac{1}{8}G_0^{jl}G_0^{ks}F_{jlksi}\right] + \ldots .\end{aligned} \tag{72}$$

The reader should note that the equations for effective action for the NJL model are analogous to those of QED, through the substitutions

$$\mu, \nu, \ldots, A_\mu \Rightarrow i, j, \ldots, \phi^i.$$

For this reason we give expressions for the effective action without carrying out the detailed calculations.

For $\tilde{\Gamma}_0$ we have

$$\frac{\delta\tilde{\Gamma}_0}{\delta\phi^i} = -D_{ij}^{-1}\phi^j + F_i \tag{73}$$

which gives us

$$\tilde{\Gamma}_0 = -\frac{1}{2}\phi^i D_{ij}^{-1}\phi^j - F = S. \tag{74}$$

Using Eq.(74) and

$$G_0^{lj}\frac{\delta^2\tilde{\Gamma}_0}{\delta\phi^j\delta\phi^k} = -\delta_k^l$$

we set (in matrix notations)

$$\hat{G}_0 = \hat{D}\left(1+\hat{F}\hat{D}\right)^{-1}, \tag{75}$$

where $\hat{F} \Rightarrow \{F_{ij}\}$ and $\hat{D} \Rightarrow \{D_{ij}\}$. Integration of the equation

$$\frac{\delta\tilde{\Gamma}_1}{\delta\phi^i} = -iT_i - \frac{i}{2}G_0^{lj}F_{jli} \tag{76}$$

gives :

$$\tilde{\Gamma}_1 = i\mathrm{Tr}\,\log\hat{K} + \frac{i}{2}\mathrm{Tr}\,\log\left(1+\hat{F}\hat{D}\right). \tag{77}$$

This is the total one–loop "effective action" with fermionic part for the NJL model.

The equation for $\tilde{\Gamma}_2$ is:

$$\frac{\delta\tilde{\Gamma}_2}{\delta\phi^i} = -\frac{i}{2}G_1^{lj}F_{jli} - \frac{1}{2}G_0^{lj}T_{jli} + \frac{1}{6}G_0^{jkl}F_{lkji} - \frac{1}{8}G_0^{jl}G_0^{ks}F_{jlksi}. \tag{78}$$

Following the steps given in Appendix A for the case of the QED we get:

$$\tilde{\Gamma}_2 = \frac{1}{2}\mathrm{Tr}\,\left(\hat{G}_0\hat{T}\right) + \frac{1}{8}G_0^{jl}G_0^{ks}F_{jlks} - \frac{1}{12}G_0^{js}G_0^{kp}G_0^{ld}F_{spd}F_{jkl}. \tag{79}$$

The graphic representations for the $\tilde{\Gamma}_2$ are the same as in QED excluding two points:

- Each waveline is correspond to the Green's function of the $\phi-$field–D_{ij} and the extra minus sign for each D_{ij}–line should be take into account;
- It is necessary to take into account that D_{ij} is a $\delta-$function, i.e., two points in diagrams connected with $\phi-$lines must be reduced to one point in fact.

8 The Two-loop Effective Potential for the NJL model in the even–dimensional space–time

We already pointed out that our expressions for $\tilde{\Gamma}$ are complicated integrals containing in integrands the classical fields $A_\mu, \bar{\psi}, \psi, \sigma, \pi$. It is possible to integrate these expressions when these classical fields are set equal to constants. In this case the effective action will be transformed to the effective potential V:

$$\Gamma = -\int dxV.$$

The effective potential was calculated for many models [8] but usually only in the one-loop approximation. In this section we would like calculate the effective potential for NJL model in the two-loop approximation.

Omitting in the formulas Eqs.(75,77,79) the parts contributing to fermion-fermion scattering the relevant part of the effective action (denoted below as $\tilde{\Gamma}_{\phi}$) for our purposes is:

$$\tilde{\Gamma}_{\phi} = \frac{1}{2}\phi^2 + i\mathrm{Tr}\,\log\hat{K} + \frac{1}{2}\mathrm{Tr}\,\left(\hat{G}_0\hat{T}\right) \tag{80}$$

(for the sake of comparison with other authors we put here $\hbar = 1$). The terms in Eq.(80) corresponds to the tree, the one-loop and the two-loop actions, respectively. To return to the ordinary notations, it is necessary to perform some calculations. Let's, for instance, rewrite the second term as follows:

$$\begin{aligned} &\mathrm{Tr}\,\log\hat{K} = \mathrm{Tr}\,\log(i\hat{\partial} + g\hat{\phi})^{-1} = \mathrm{Tr}\,\log((i\hat{\partial})^{-1}(1 + g\hat{\phi}(i\hat{\partial})^{-1})^{-1}) = \\ &= \mathrm{Tr}\,\log\hat{K}_0 - \mathrm{Tr}\,\log(1 + g\hat{\phi}\hat{K}_0) \end{aligned} \tag{81}$$

and as is usually done neglect the first term (because it gives us a trivial infinite contribution). As shown in Appendix B for the one -loop effective potential we have

$$V_1 = i\mathrm{Tr}\,\log(1 + g\hat{\phi}\hat{K}_0) = i2^{\frac{d}{2}-1}\int\frac{d^dp}{(2\pi)^d}\log\left(1 - \frac{g^2\phi^2}{p^2}\right), \tag{82}$$

where d is the dimension of the space-time.

For $d = 2$ we get the following function

$$\begin{aligned} V_1 &= -\frac{1}{4\pi}\Lambda^2\log(1 + g^2\phi^2/\Lambda^2) - \frac{g^2\phi^2}{4\pi}\log(1 + \Lambda^2/g^2\phi^2) \simeq \\ &\simeq -\frac{g^2\phi^2}{4\pi}\left[1 - \log(g^2\phi^2/\Lambda^2)\right], \end{aligned} \tag{83}$$

where Λ^2 is the cut-off parameter in the momentum space. The NJL model in two dimensional space–time is renormalizable. We may renormalize our effective potential V in the two-dimensional space–time by demanding [1, 8]:

$$\frac{\partial^2 V}{\partial\phi^2}|_{\phi_0} = 1,$$

where ϕ_0^2 is arbitrary subtraction point. Hence, the renormalised one–loop effective potential is:

$$V_r = \frac{1}{2}\phi^2 + \frac{g^2\phi^2}{4\pi}\left(\log(\phi^2/\phi_0^2) - 3\right).$$

For $d=4$ we get from Eq.(80) the following one-loop formula:

$$
\begin{aligned}
V_1 &= -\frac{\Lambda^4}{16\pi^2}\ln\left(1+\frac{g^2\phi^2}{\Lambda^2}\right) - \frac{1}{16\pi^2}g^2\phi^2\Lambda^2 + \frac{g^4\phi^4}{16\pi^2}\log\left(1+\Lambda^2/g^2\phi^2\right) \simeq \\
&\simeq -\frac{1}{8\pi^2}g^2\phi^2\Lambda^2 + \frac{g^4\phi^4}{16\pi^2}\left(\frac{1}{2}+\log(\Lambda^2/g^2\phi^2)\right).
\end{aligned} \tag{84}
$$

Let's now turn to the calculation of the two-loop contribution to the effective potential.

In the following expression

$$
\begin{aligned}
&\tfrac{1}{2}\mathrm{Tr}\,\left(\hat{G}_0\hat{T}\right) = \tfrac{1}{2}\left[\hat{D}(1+\hat{F}\hat{D})\right]^{ij}\mathrm{Tr}\,(\gamma_j\hat{K}\gamma_i\hat{K}) = \\
&= \tfrac{1}{2}g^2\mathrm{Tr}\,(\gamma_j\hat{K}\gamma_i\hat{K})\delta^{ij} + \text{fermionic} \quad \text{parts},
\end{aligned} \tag{85}
$$

we consider only the first term and write it as:

$$
\int dx V_2 = -\frac{1}{2}g^2\mathrm{Tr}\,(\gamma_j\hat{K}\gamma_i\hat{K})\delta^{ij} = -\frac{1}{2}g^2\int dx\mathrm{Tr}\,\left(\gamma_i\hat{K}(0)\gamma^i\hat{K}(0)\right). \tag{86}
$$

It is easy to see, that

$$
\hat{K}(0) = \int[d^dp/(2\pi)^d](\hat{p}+g\hat{\phi})^{-1} = \sum_{n=0}^{\infty}(-g)^n\int[d^dp/(2\pi)^d]\hat{p}(\hat{\phi}\hat{p})^n(p^2)^{-n-1}. \tag{87}
$$

Taking into account the vanishing of the contributions of the terms with even n and

$$
\hat{\phi}\hat{p}\hat{\phi}\hat{p} = \phi^2p^2
$$

we derive

$$
\hat{K}(0) = -g\int[d^dp/(2\pi)^d]\hat{p}\hat{\phi}\hat{p}(p^2-g^2\phi^2)^{-1}(p^2)^{-1} \tag{88}
$$

The substitution of Eq.(88) into Eq.(86) gives us

$$
\begin{aligned}
&V_2 = -\frac{1}{2}g^4\int[d^dp/(2\pi)^d]\int[d^dq/(2\pi)^d](p^2-g^2\phi^2)^{-1}(q^2-g^2\phi^2)^{-1}\cdot \\
&\cdot\mathrm{Tr}\,\left(\gamma_i\hat{p}\hat{\phi}\hat{p}\gamma_i\hat{q}\hat{\phi}\hat{q}\right).
\end{aligned} \tag{89}
$$

Of course, one must understand integrals (87), (88), (89) as regularized, for example, by a cut-off parameter Λ^2. Using the following identity:

$$
\hat{p}\hat{\phi}\hat{p} = (\sigma - i\gamma_5\pi)p^2
$$

we see that

$$\gamma_i \hat{p}\hat{\phi}\hat{p}\gamma_i \hat{q}\hat{\phi}\hat{q} = \gamma_i(\sigma - i\gamma_5\pi)\gamma_i(\sigma - i\gamma_5\pi)p^2q^2 = \\ = p^2q^2[(\sigma - i\gamma_5\pi) + \sigma(i\gamma_5)^2 - i\gamma_5(i\gamma_5)^2\pi](\sigma - i\gamma_5\pi) = 0. \tag{90}$$

Thus we see that the two-loop contribution to the effective potential in the NJL model vanishes and this result does not depend on the dimension of space-time.

Concerning NJL model in four space-time dimension we see that as indicated in the earlier work of Nambu and Jona-Lasinio there is a relationship between g^2 and the cut-off parameter Λ^2 (in the one-loop approximation)

$$g^2\Lambda^2 > 4\pi^2, \tag{91}$$

whose validity is crucial for the dynamical generation of fermion mass. This relation can be made more precise. With this ultimate aim let us rewrite the four dimensional one-loop effective potential as follows

$$V = \left(\frac{\Lambda^2}{2g^2} - \frac{\Lambda^4}{8\pi^2}\right)x + \frac{\Lambda^4}{16\pi^2}x^2\left(\frac{1}{2} - \log x\right) \tag{92}$$

where we denoted $x = g^2\phi^2/\Lambda^2$. The minimum of this function is obtained at

$$x_M \log x_M = -1 + 4\pi^2/(g^2\Lambda^2) \tag{93}$$

which by taking into account of $x < 1$ (remembering, that Λ^2 is very large) gives us Eq.(91). But for the second derivative of V we have:

$$V" = -\frac{\Lambda^4}{8\pi^2}(\log x + 1),$$

from which we see that for positivity at the point x_M we must have

$$4\pi^2/(g^2\Lambda^2) < 1 - x_M. \tag{94}$$

Acknowledgements

One of the authors (B.F.) would like to thank Prof. Namik K.Pak for warm hospitality at TÜBITAK and at the Midlle East Technical University, Turkey. We also are gratefule to Prof. Namik K.Pak for discussions , careful reading and corrections of the manuscript .

This work was supported in part by the Foundation for Fundamental Investigations of Uzbekistan State Committee for Science and Technics under contract N 40, International Science Foundation Grant MZJ000 and by DOPROG Program, TÜBITAK, Turkey.

A Solving of the Eq.(60)

For making a transition from Eq.(60) to Eq.(61) we need to calculate G_1 from the second of Eqs.(56):

$$G_1^{\mu\nu} = G_0^{\mu\sigma} \frac{\delta^2 \tilde{\Gamma}_1}{\delta A^\sigma \delta A^\lambda} G_0^{\lambda\nu}$$

which gives us

$$G_1^{\mu\nu} = G_0^{\mu\sigma} \left(iT_{\sigma\lambda} - \frac{i}{2} \frac{\delta}{\delta A^\sigma} \left(G_0^{\alpha\beta} F_{\beta\alpha}^{\lambda} \right) \right) G_0^{\lambda\nu}.$$

Substituting the last equation into Eq.(60) we have

$$\frac{\delta \tilde{\Gamma}_2}{\delta A_\mu} = -\frac{i}{2} G_0^{\mu\sigma} \left(iT_{\sigma\lambda} - \frac{i}{2} \frac{\delta}{\delta A^\sigma} \left(G_0^{\alpha\beta} F_{\beta\alpha}^{\lambda} \right) \right) G_0^{\lambda\nu} F_{\sigma\lambda}^{\mu} -$$

$$+\frac{1}{2} G_0^{\sigma\lambda} T_{\sigma\lambda}^{\mu} + \frac{1}{6} G_0^{\sigma\lambda\nu} F_{\sigma\lambda\nu}^{\mu} - \frac{1}{8} G_0^{\sigma\lambda} G_0^{\nu\rho} F_{\sigma\lambda\nu\rho}^{\mu}$$

The first and third terms of this relation can be combined as follows

$$-\frac{i}{2} G_0^{\mu\sigma} iT_{\sigma\lambda} G_0^{\lambda\nu} F_{\sigma\lambda}^{\mu} - \frac{1}{2} G_0^{\sigma\lambda} T_{\sigma\lambda}^{\mu} = \frac{1}{2} \frac{\delta}{\delta A_\mu} \mathrm{Tr} \left(\hat{G}_0 \hat{T} \right) \qquad (A.1)$$

where we have used the matrix notations: $\hat{G} = \{G_{\mu\nu}\}, \hat{T} = \{T_{\mu\nu}\}$. Using for simplicity the notation $\frac{\delta}{\delta A_\mu} \Rightarrow \delta_\mu$ we have for remaining part of $\delta\tilde{\Gamma}_2/\delta A_\mu$:

$$-\tfrac{1}{4} G_0^{\sigma\sigma_1} \delta_{\sigma_1} \left(G_0^{\alpha\beta} \delta^{\lambda_1} F_{\beta\alpha} \right) G_0^{\lambda_1\lambda} \delta^\mu F_{\sigma\lambda} - \tfrac{1}{6} G_0^{\alpha\lambda} G_0^{\beta\nu} G_0^{\gamma\sigma} \delta^\mu F_{\alpha\beta\gamma} F_{\lambda\nu\sigma} +$$

$$+\tfrac{1}{8} G_0^{\alpha\beta} G_0^{\gamma\delta} \delta^\mu F_{\alpha\beta\gamma\delta}.$$

After some manipulations we may transform the last expression into the following form

$$\frac{\delta}{\delta A_\mu} \left[\frac{1}{8} G_0^{\alpha\beta} G_0^{\gamma\delta} F_{\alpha\beta\gamma\delta} - \frac{1}{12} G_0^{\alpha\sigma} G_0^{\beta\nu} G_0^{\gamma\lambda} F_{\alpha\beta\gamma} F_{\sigma\nu\lambda} \right] \qquad (A.2)$$

The sum of Eqs.(A.1) and (A.2) leads to the Eq.(61).

B One–loop effective potential for the NJL model

In this appendix we would like to compute the one-loop effective potential for NJL model. All the steps of the calculation is familiar and we present it only for complete-

ness. The following chain of equalities gives us the Eq.(82):

$$\int dxV_1 = i\mathrm{Tr}\,\log(1+g\hat{\phi}\hat{K}_0) = i\sum_{n=1}^{\infty}\frac{g^n}{n}(-1)^{n-1}\int dx_1\cdots dx_n\cdot$$

$$\cdot\mathrm{Tr}\,\left(\hat{\phi}\hat{K}_0(x_1-x_2)\hat{\phi}\hat{K}_0(x_2-x_3)\cdots\hat{\phi}\hat{K}_0(x_n-x_1)\right) =$$

$$= i\sum_{n=1}^{\infty}\frac{g^n}{n}(-1)^{n-1}\int dxdp\mathrm{Tr}\,\left(\hat{\phi}\hat{p}^{-1}\hat{\phi}\hat{p}^{-1}\cdots\hat{\phi}\hat{p}^{-1}\right) =$$

$$= i\sum_{n=1}^{\infty}\frac{g^n}{n}(-1)^{n-1}\int dxdpp^{-2n}\phi_{i_1}\phi_{i_2}\cdots\phi_{i_n}\mathrm{Tr}\,\left(\gamma_{i_1}\hat{p}\gamma_{i_2}\hat{p}\cdots\gamma_{i_n}\hat{p}\right) =$$

$$= i2^{\frac{d}{2}}\int dxdp\sum_{n=1}^{\infty}\frac{g^{2n}}{2n}(-1)^{2n-1}(\phi^2/p^2)^n =$$

$$= i2^{\frac{d}{2}-1}\int dxdp\log(1-g^2\phi^2/p^2).$$

References

[1] S.Coleman and E.Weinberg, *Phys.Rev.***D7** (1973) 1888.

[2] R.Jackiw, *Phys.Rev.***D9** (1974) 1686.

[3] J.Iliopoulos ,C.Itzykson and A.Martin, *Rev.Mod.Phys.***47** (1975) N1,165.

[4] B.A.Faizullaev and M.M.Musakhanov, *Turk.J.of Phys.***17**(1993) 717.

[5] B.S.DeWitt, Dynamical Theory of Groups and Fields (Gordon and Breach, N.-Y.) 1965.

[6] Y.Nambu and G.Jona-Lasinio, *Phys.Rev.***122**(1961) 345.

[7] D.Gross and A.Neveu, *Phys.Rev.***D10**(1974) 3235.

[8] V.P. Gusynin and V.A.Miranskii, *Journ.Eksp.i Teor.Fiz.*, **101**, (1992) 414.

Figures Captions

Figure 1. The two–loop contribution to the effective action in the $\lambda\varphi^4$ theory.

Figure 2. Diagrams which corresponds to the term $\mathrm{Tr}\,(\hat{G}_0\hat{T})$ in the $\tilde{\Gamma}_2$.

Figure 3. Diagrams which corresponds to the term $G_0^{\alpha\beta}G_0^{\gamma\delta}F_{\alpha\beta\gamma\delta}$ in the $\tilde{\Gamma}_2$.

Figure 4. Diagrams which corresponds to the term $G_0^{\alpha\sigma}G_0^{\beta\nu}G_0^{\gamma\lambda}F_{\alpha\beta\gamma}F_{\sigma\nu\lambda}$ in the $\tilde{\Gamma}_2$.

a)

b)

Fig.1

a)

b)

Fig.2

Fig.3

Fig.4